Discrete Mathematics: Combinatorics and Graph Theory

Practice Exam 2

Instructions. Solve any 5 questions and state which 5 you would like graded. Note that this is a sample exam, and while it bears some similarity to the real exam, the two are not isomorphic.

1. Consider the following:
   (a) Define a relation R on $\mathbb{Z}$ by $(a, b) \in R$ if $a^2 - b^2 \leq 3$. Verify whether R is (i) reflexive (ii) symmetric and (iii) transitive?
   (b) Let $A = \{1, 2, 3, 4, 5\}$ and
   $$R = \{(1,1),(1,3),(1,4),(2,2),(2,5),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(5,2),(5,5)\}.$$
   Which of the following is an equivalence class?
   $(i)\quad \{1,2,3\} \qquad (ii)\quad \{2,3,5\} \qquad (iii)\quad \{1,3,4\} \qquad (iv)\quad \{1,2\} \qquad (v)\quad \{1,2,3,4,5\}$
   (c) If R_1 and R_2 are equivalence relations on the set A, then $R_1 \cap R_2$ is an equivalence relation on A. Prove or disprove.

2. Solve the following:
   (a) $3x \equiv 7 \pmod 4$
   (b) $6x \equiv 7 \pmod 8$
   (c) $8x \equiv 13 \pmod{29}$

3. Find the smallest positive integer x such that:
$$x \equiv 1 \pmod 3$$
$$x \equiv 2 \pmod 4$$
$$x \equiv 3 \pmod 5$$

4. Evaluate the following expressions or verify the identities:
   (a) $(a+b)^7$
   (b) $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$
   (c) $\binom{n}{k} = \binom{n}{n-k}$
   (d) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
   (e) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$

5. Let $gcd(a, b) = 1$. Show that $a^{\phi(b)} + b^{\phi(a)} \equiv 1$ (mod ab).

6. Let p be a prime. Show that $\binom{p}{i} \equiv 0 \pmod p$.